Electrical and Electronics Engineering
2024-2025
Master Semester 2

Course
Smart grids technologies

DFT-based Synchrophasor Estimation Algorithms – The Interpolated DFT (IpDFT)

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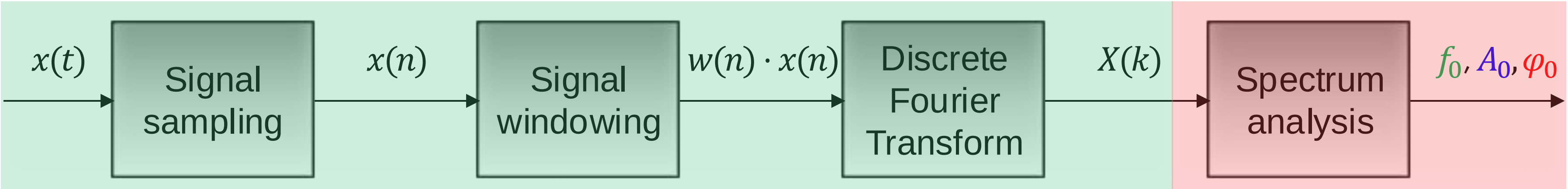
Recall



Topics addressed during last lectures and laboratory sessions:

1. Signal sampling:
   - Analytical derivation of aliasing and Nyquist-Shannon theorem.
2. The Discrete Fourier Transform:
   - Analytical derivation of spectral leakage and special windowing functions.

Topics to be addressed in this lecture and laboratory session:

3. Identification of the main tone parameters via Interpolated DFT (IpDFT):
   - Estimation of the main tone frequency f_0, amplitude A_0, phase φ_0.

A trivial DFT-based synchrophasor estimator

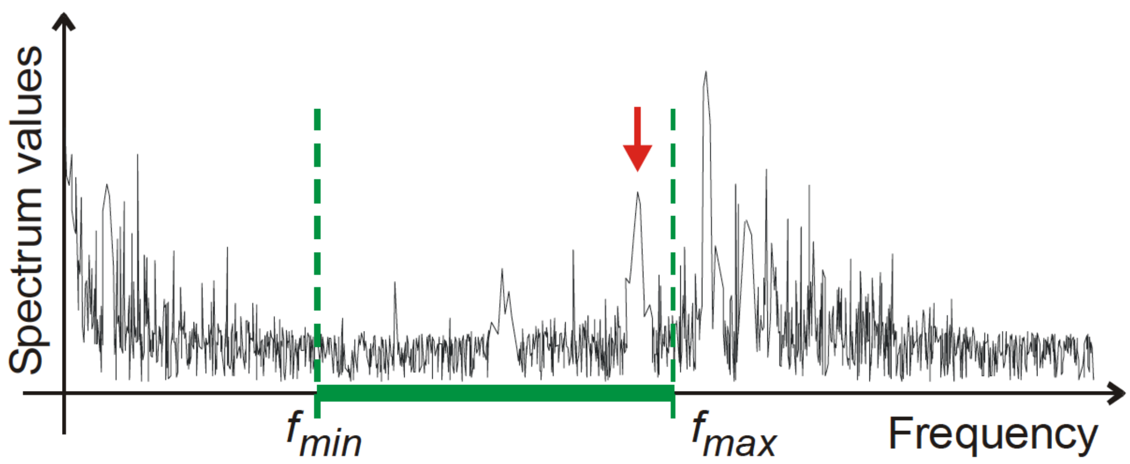
The main task of a synchrophasor estimation algorithm is to assess the parameters of the fundamental tone of a signal by using a previously acquired set of samples representing a portion of an acquired waveform (i.e., node voltage and/or branch/nodal current).

A trivial approach to estimate the parameters of the main DFT tone might be based on the estimation of the position of a local DFT maximum (i.e., say the k_m bin of the spectrum) within a specific frequency range. Based on this approach the synchrophasor estimated parameters f_0, A_0 and $\varphi_0$may be computed as follows:

$$\hat{f}_0 = k_m \Delta f$$

$$\hat{A}_0 = |X(k_m)|$$

$$\hat{\varphi}_0 = \angle X(k_m)$$



Introduction

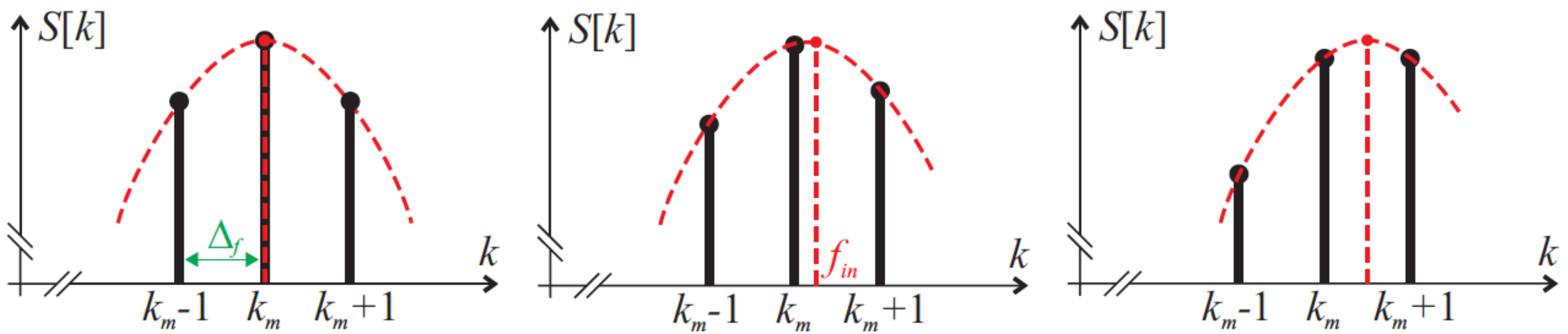
A trivial DFT-based synchrophasor estimator

The accuracy of this trivial synchrophasor estimator is related to the accuracy in the location of the DFT peak. In particular the maximum error in the peak location is equal to half of the DFT frequency resolution $\Delta f = 1/T$, as the main spectrum tone may lie somewhere between the highest and 2nd highest bin of the DFT spectrum.

In particular it should be noticed that the relative error in the frequency estimation

$$\max(\varepsilon_{f,r}) = \frac{\max(\varepsilon_f)}{f_0} = \frac{\max|k_m \Delta f - f_0|}{f_0} = \frac{1}{2}\frac{\Delta f}{f_0} = \frac{1}{2T f_0} = \frac{F_s}{2N f_0}$$

Is (unfortunately) maximized when $F_S >> f_0$ (i.e. for components lying in the beginning of the signal spectrum), that is the case of typical synchrophasor estimation algorithms.



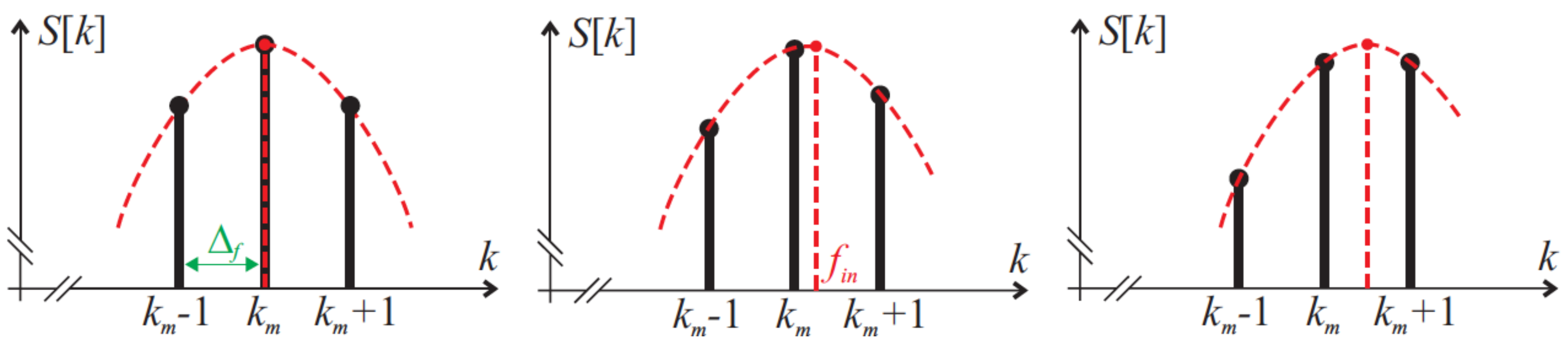
DFT interpolation

In order to improve the accuracy of this trivial synchrophasor estimator there are several options:

1. **Decrease the sampling frequency** (see previous formula)
   CONS: aliasing may arise

2. **Increase the window length** (i.e., improve the frequency resolution)
   CONS: higher number of samples to be processed ➔ higher computation time
   CONS: in order to reach high accuracy levels, very long windows are needed

3. **DFT interpolation**
   A more exact estimation of the main spectrum tone location can be given by calculating the abscissa of the maximum of an interpolation curve of the DFT spectrum



IpDFT formulation

Signal model, time-windowing and DFT

Let's consider a very simple discrete-time signal $x(n)$ produced by a sampling process characterized by a sampling frequency F_S

$$x(n) = A_0 \cos(2\pi f_0 n T_s + \varphi_0)$$

Being f_0, A_0 and φ_0 the signal frequency, amplitude and phase respectively. As we have seen, the input signal is sliced in portions containing N samples using a pre-selected windowing function $w(n)$; its DFT spectrum can be computed:

$$X(k) \triangleq \frac{2}{B} \sum_{n=0}^{N-1} w(n) x(n) W_N^{kn}$$

As known, if the window does not contain an integer number of periods $k(1/f_0), k \in \mathbb{N}$, of the signal $x(n)$, leakage occurs. As a consequence, **the main tone of the signal is located between two consecutives DFT bins**. Its location can therefore be expressed as follows:

$$k_{peak} = k_m + \delta$$

being k_m **the index of the DFT bin characterized by the highest magnitude** and $-0.5 < \delta < 0.5$ a **fractional correction term**.

IpDFT formulation

An intuitive interpretation

From the last equation, the IpDFT problem may be formulated as follows:

Based on the DFT spectrum $X(k)$ of the signal $x(n)$ analyzed with the known windowing function $w(n)$, find the correction term δ that better approximates the exact location of the main spectrum tone.

To be noticed that k_{peak} can also be interpreted as the number of acquired cycles of the input signal. Therefore if:

$$k_{peak} = k_m \ (\delta = 0)$$

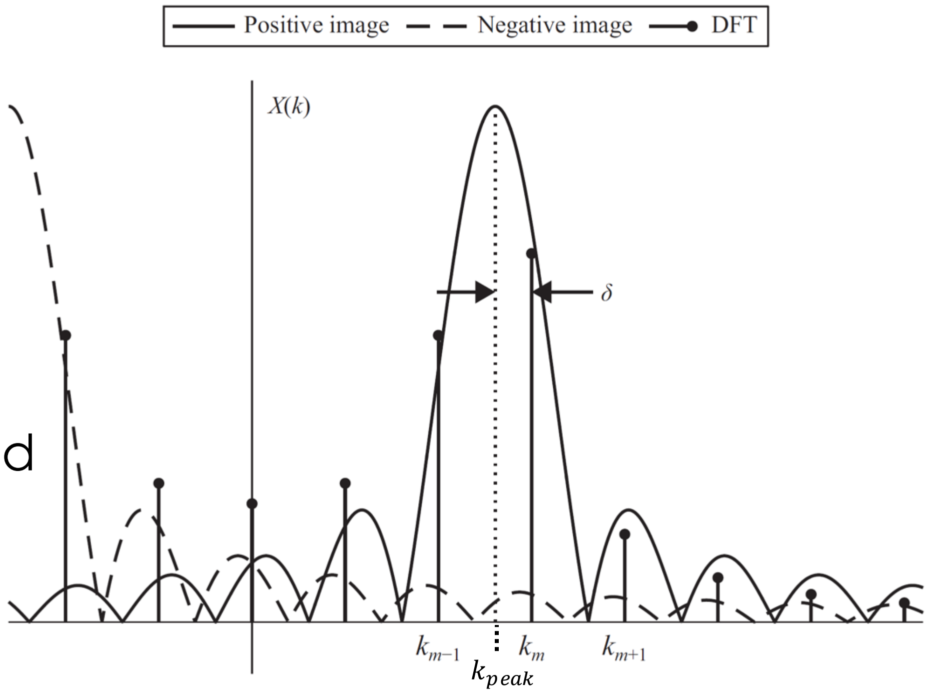
↓

coherent sampling (NO leakage)

$$k_{peak} \neq k_m \ (\delta \neq 0)$$

↓

incoherent sampling (LEAKAGE)



Based on the number of DFT bins used to perform the interpolation, IpDFT algorithms may be separated in 2-points, 3-points, … DFT interpolators.

IpDFT formulation

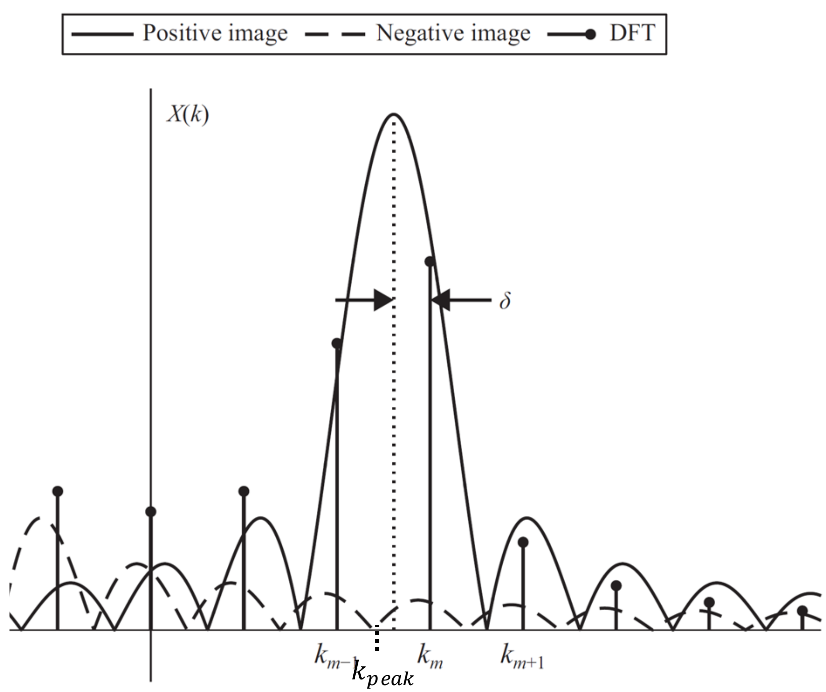
Analytical formulation

The spectrum of $x(n)$ can be expressed in terms of the positive and negative images of the main frequency tone at the unknown frequency f_0 (or ω_0):

$$X(\omega) = \frac{A_0}{2} e^{j\varphi_0} W(\omega - \omega_0) + \frac{A_0}{2} e^{-j\varphi_0} W(\omega + \omega_0)$$

being $W(\omega)$ the Fourier transform of the selected window function. **Assuming that the effects of leakage are properly compensated by windowing, we can neglect the long-range spectral leakage produced by the negative spectrum image and express the 2 highest bins of the DFT** $X(k_m)$, the highest, and $X(k_m + \varepsilon)$, the second highest, with $\varepsilon = \pm 1$ **as a function of the positive spectrum image only:**

$$\frac{|X(k_m+\varepsilon)|}{|X(k_m)|} = \frac{|X(\omega_{m+\varepsilon})|}{|X(\omega_m)|} = \frac{|W(\omega_{m+\varepsilon} - \omega_0)|}{|W(\omega_m - \omega_0)|}$$
$$= \frac{|W(\omega_0 + (\varepsilon - \delta) \cdot 2\pi F_s/N - \omega_0)|}{|W(\omega_0 - \delta \cdot 2\pi F_s/N - \omega_0)|}$$
$$= \frac{|W(\,(\varepsilon - \delta) \cdot 2\pi F_s/N)|}{|W(-\delta \cdot 2\pi/NF_s)|}$$



Note that the **DFT frequency resolution** is $\Delta\omega = 2\pi\Delta f = \frac{2\pi}{T} = 2\pi F_s/N$ and δ is unknown.

IpDFT formulation

Solution for the rectangular window

Let us recall from lecture 1.3 the Fourier transform of the rectangular windowed signal:

$$\Im\{w_R(t)\cdot x(t)\} = \Im\{w_R(t)\cdot A_0\cos(2\pi f_0 t+\varphi_0)\} = W_R(f) * X(f)$$
$$= \mathrm{A_0 e}^{j\varphi_0} T\,\mathrm{sinc}(fT) * \frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)] = \mathrm{A_0 e}^{j\varphi_0}\frac{T}{2}\left[\mathrm{sinc}\big((f-f_0)T\big)+\mathrm{sinc}\big((f+f_0)T\big)\right]$$

Let us assume (again) that **the effects of leakage are properly compensated by windowing, so we can neglect the long-range spectral leakage produced by the negative image**. Therefore, we can say

$$\Im\{w_R(t)\cdot x(t)\} = \mathrm{A_0 e}^{j\varphi_0}\frac{T}{2}\mathrm{sinc}\big((f-f_0)T\big)$$

Therefore, we have that (recall that $T = N/F_s$):

$$\mathrm{A_0 e}^{j\varphi_0}\frac{T}{2}\mathrm{sinc}\big((f-f_0)T\big) = \mathrm{A_0 e}^{j\varphi_0} W_R(\omega-\omega_0) = \mathrm{A_0 e}^{j\varphi_0}\frac{N}{2F_s}\frac{\sin\left(\frac{\omega-\omega_0}{2}\frac{N}{F_s}\right)}{\frac{\omega-\omega_0}{2}\frac{N}{F_s}} = \frac{\sin\left(\frac{\omega-\omega_0}{2}\frac{N}{F_s}\right)}{\omega-\omega_0}$$

Let us now express the ratio $\frac{|W(\,(\varepsilon-\delta)\cdot 2\pi F_s/N)|}{|W(-\delta\cdot 2\pi/NF_s)|}$ previously obtained by using the $W_R(\omega)$.

IpDFT formulation

Solution for the rectangular window

We obtain the following for the rectangular window:

$$\frac{|W_R(\,(\varepsilon-\delta)\cdot 2\pi F_S/N)|}{|W_R(-\delta\cdot 2\pi/NF_S)|} = \left|\frac{\sin\left(\frac{(\varepsilon-\delta)\cdot 2\pi F_S/N)}{2}\frac{N}{F_S}\right)}{(\varepsilon-\delta)\cdot 2\pi F_S/N)}\right| \left|\frac{-\delta\cdot 2\pi F_S/N}{\sin\left(\frac{-\delta\cdot 2\pi F_S/N)}{2}\frac{N}{F_S}\right)}\right|$$

$$\left|\frac{\sin(\pi(\varepsilon-\delta))}{(\varepsilon-\delta)}\right| \left|\frac{-\delta}{\sin(-\delta\pi)}\right| =$$

$$=\left|\frac{-\sin(-\delta\pi)}{(\varepsilon-\delta)}\right| \quad \left|\frac{-\delta}{\sin(-\delta\pi)}\right| = \left|\frac{\delta}{\delta-\varepsilon}\right|$$

In view of the above, we can estimate δ (the only unknown) as:

$$\hat{\delta} = \varepsilon\frac{|X(k_m+\varepsilon)|}{|X(k_m)|+|X(k_m+\varepsilon)|}$$

and the signal frequency is then:

$$\hat{f}_0 = (k_m+\hat{\delta})\Delta f$$

IpDFT formulation

Solution for the rectangular window

To compute the signal magnitude $\hat{A}_0$, we may recall that we expressed the Fourier transform of the windowed signal as:

$$\Im\{w_R(t) \cdot x(t)\} = \mathrm{A}_0 \mathrm{e}^{j\varphi_0} \frac{T}{2} \operatorname{sinc}\big((f - f_0)T\big)$$

Therefore, we can express the following ratio:

$$\frac{\hat{A}_0}{|X(k_m)|} = \frac{|X(k_m + \delta)|}{|X(k_m)|} = \frac{N}{2F_s} \left| \frac{-\delta \cdot 2\pi F_s/N}{\sin\left(\frac{-\delta \cdot 2\pi F_s/N)}{2} \frac{N}{F_s}\right)} \right| =$$

$$= \left| \frac{-\delta\pi}{\sin(-\delta\pi)} \right|$$

and solve for $\hat{A}_0$ to get

$$\hat{A}_0 = |X(k_m)| \left| \frac{\pi\hat{\delta}}{\sin(\pi\hat{\delta})} \right|$$

The estimated phase can be simply expressed as

$$\hat{\varphi}_0 = \angle X(k_m) - \pi\hat{\delta}$$

IpDFT formulation

Solution for Hanning (Hann) window

As known the spectrum of the discrete-time Hanning window of length N may be represented as the following combination of "*Dirichlet kernel*":

$$D_N(\omega) = e^{-j\omega(N-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

$$W_H(\omega) = -0.25 \cdot D_N(\omega - 2\pi/N) + 0.5 \cdot D_N(\omega) - 0.25 \cdot D_N(\omega + 2\pi/N)$$

By replacing in the previously defined 2-bins ratio $W(\omega) = W_H(\omega)$:

$$\frac{|X(k_m + \varepsilon)|}{|X(k_m)|} = \frac{|W_H((\varepsilon - \delta) \cdot 2\pi/N)|}{|W_H(-\delta \cdot 2\pi/N)|} \simeq \left|\frac{\delta + \varepsilon}{\delta - 2\varepsilon}\right|$$

And, in view of the above approximated equality, we can express δ as:

$$\hat{\delta} = \varepsilon \frac{2|X(k_m + \varepsilon)| - |X(k_m)|}{|X(k_m)| + |X(k_m + \varepsilon)|}$$

and the signal parameters as:

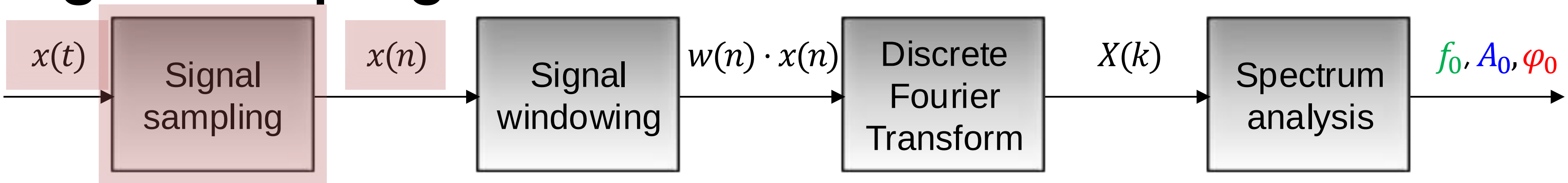
$$\hat{f}_0 = (k_m + \hat{\delta})\Delta f$$

$$\hat{A}_0 = |X(k_m)| \left|\frac{\pi\hat{\delta}}{\sin(\pi\hat{\delta})}\right| \left|\hat{\delta}^2 - 1\right|$$

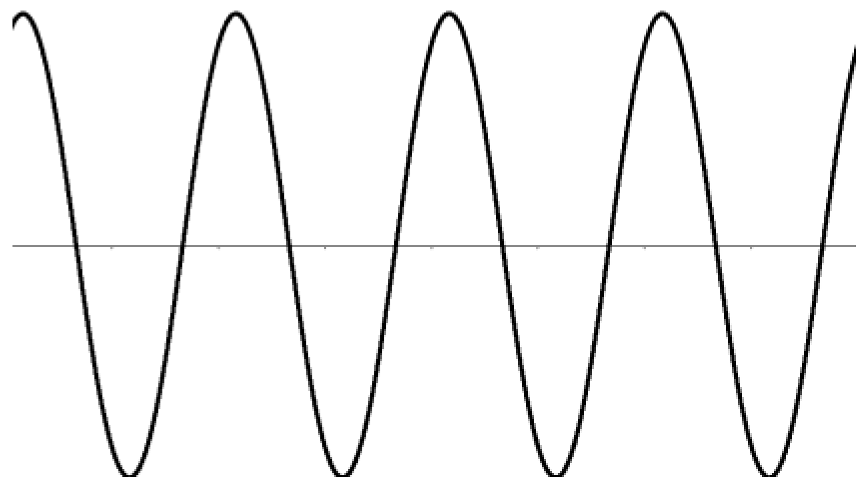
$$\hat{\varphi}_0 = \angle X(k_m) - \pi\hat{\delta}$$

IpDFT-based synchrophasor estimation

Signal sampling

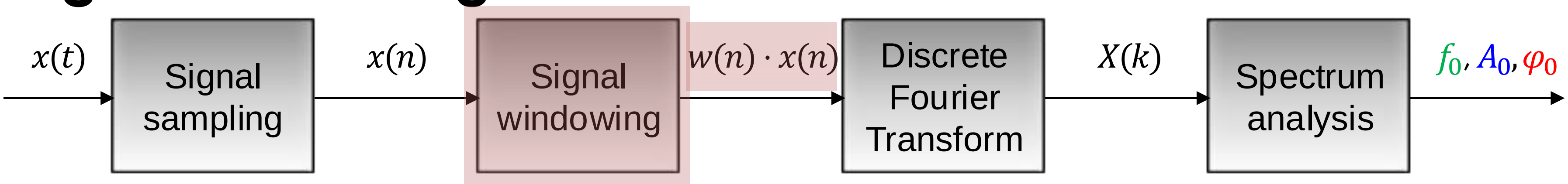


$$x(n) = A_0 \cos(2\pi f_0 n T_S + \varphi_0)$$



IpDFT-based synchrophasor estimation

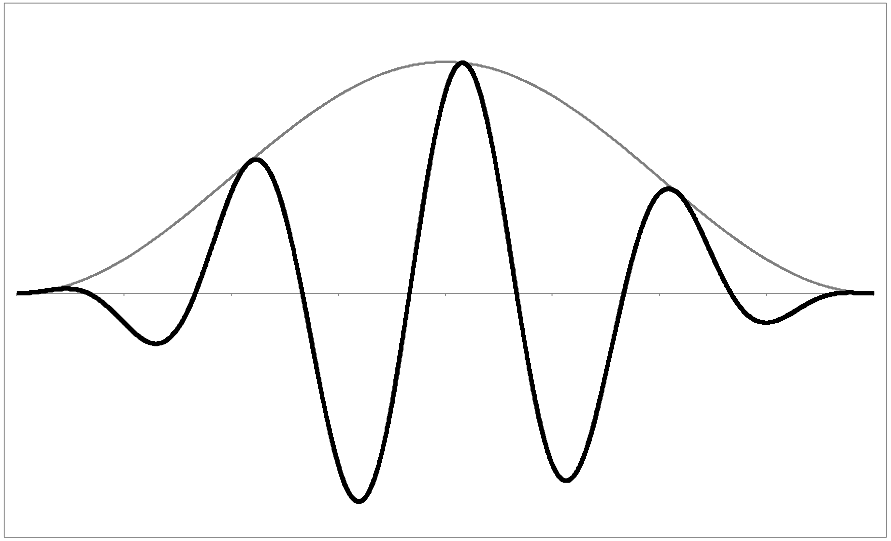
Signal windowing



Hanning window:

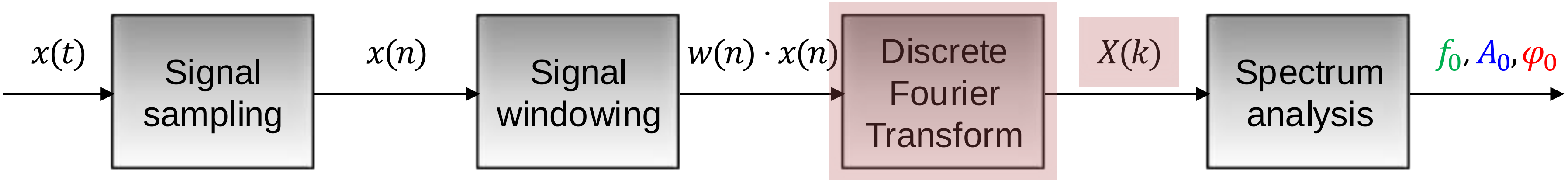
$$w_H(n) = \frac{1 - \cos\left(\frac{2\pi n}{N}\right)}{2}$$

$$n \in [0, N-1]$$



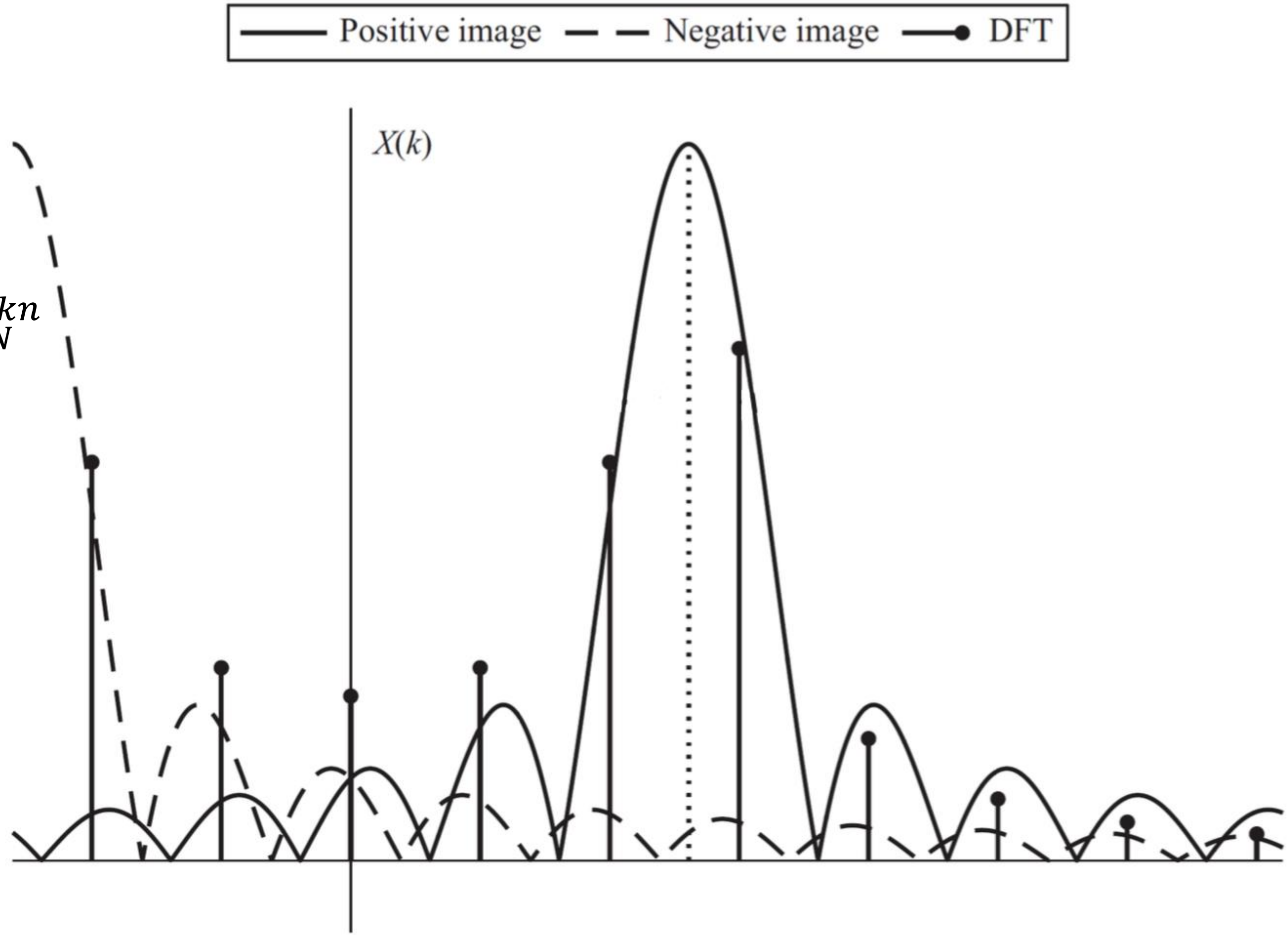
IpDFT-based synchrophasor estimation

Discrete Fourier Transform



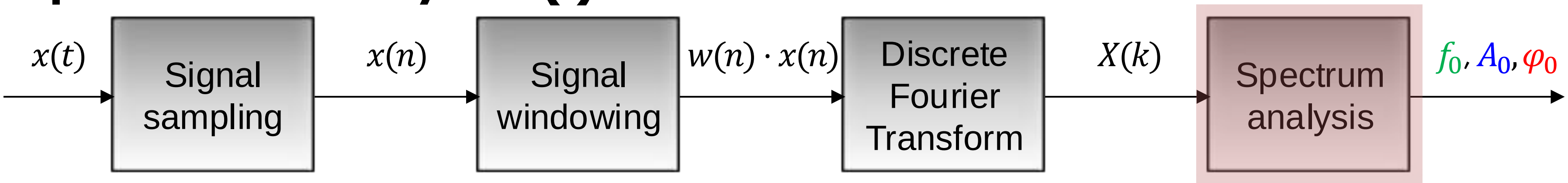
$$X(k) \triangleq \frac{2}{B} \sum_{n=0}^{N-1} w(n)x(n)W_N^{kn}$$

$$k \in [0, N-1]$$



IpDFT-based synchrophasor estimation

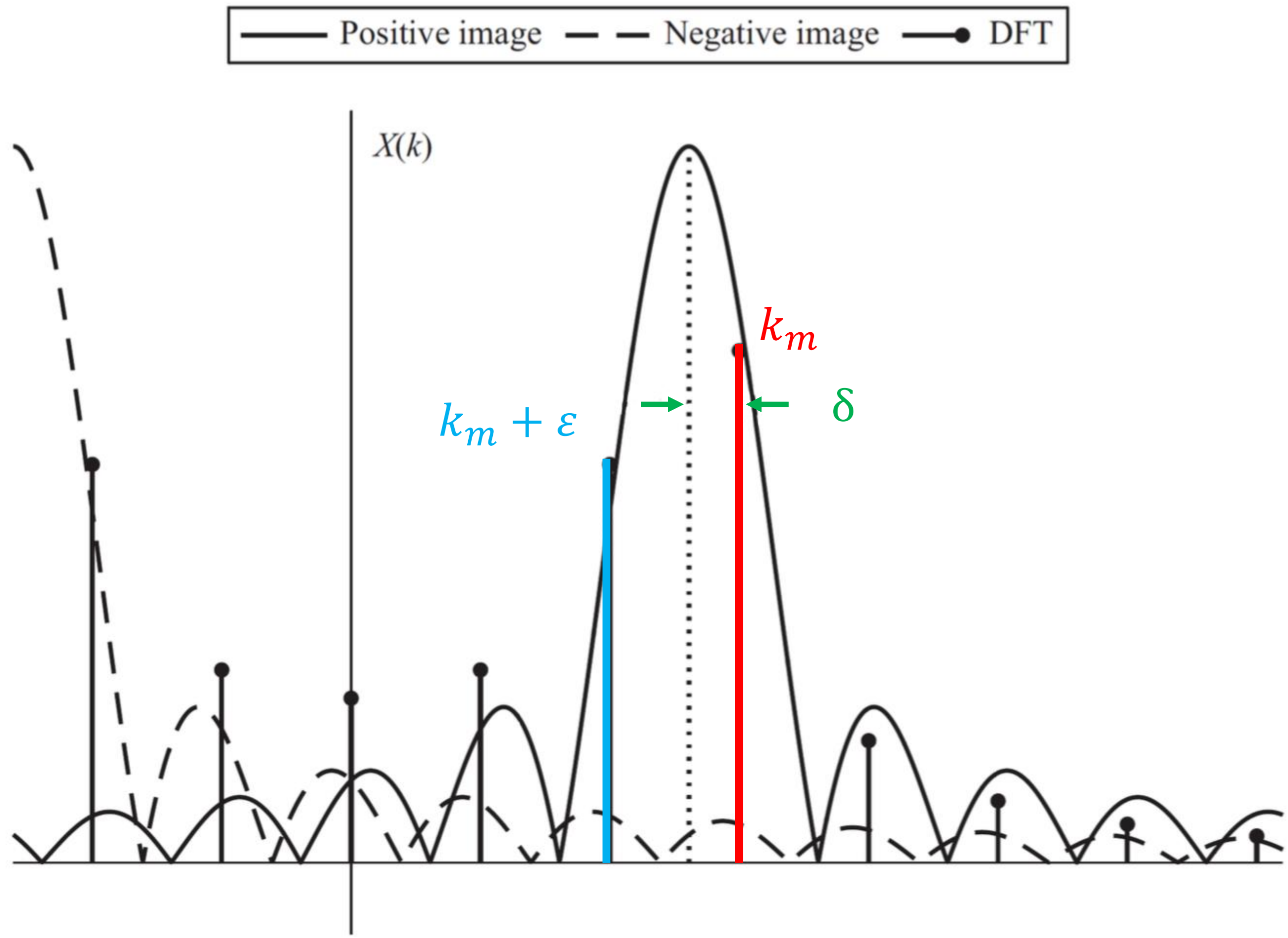
Spectrum analysis: (i) DFT maximum identification



$$|X(k_m+1)| < |X(k_m-1)|$$

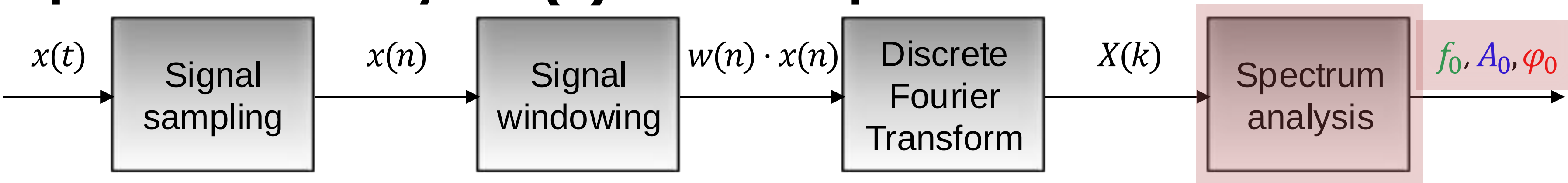
$$\delta < 0$$

$$\varepsilon = -1$$



IpDFT-based synchrophasor estimation

Spectrum analysis: (ii) DFT interpolation

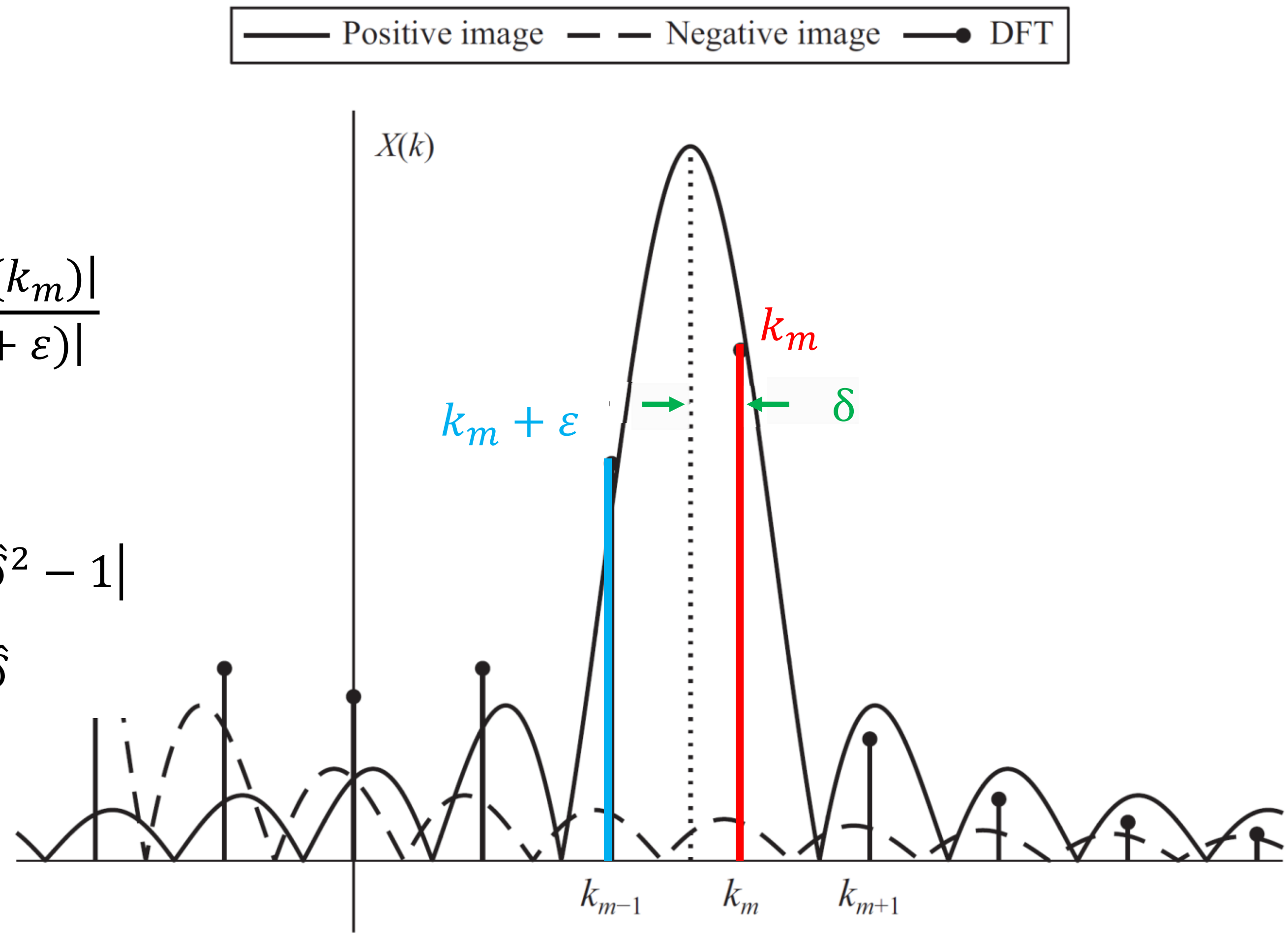


$$\hat{\delta} = \varepsilon \frac{2|X(k_m + \varepsilon)| - |X(k_m)|}{|X(k_m)| + |X(k_m + \varepsilon)|}$$

$$\hat{f}_0 = (k_m + \hat{\delta})\Delta f$$

$$\hat{A}_0 = |X(k_m)| \left| \frac{\pi\hat{\delta}}{\sin(\pi\hat{\delta})} \right| |\hat{\delta}^2 - 1|$$

$$\hat{\varphi}_0 = \angle X(k_m) - \pi\hat{\delta}$$



References

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